ELECTROMAGNETIC FIELD TENSOR AND LORENTZ FORCE AS CONSEQUENCE OF THE GEOMETRY OF MINKOWSKIAN SPACETIME

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Abstract We show that the electromagnetic field tensor and the Lorentz Force are both a natural consequence of the geometric structure of Minkowskian space, being related to infinitesimal boost and rotations in spacetime. The longstanding issue about the apparent empirical origin of the Lorentz Force is clarified.

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I. INTRODUCTION

It is a well known fact that the electromagnetic force acting on a charged particle cannot be obtained from Maxwell Equations having, ultimately, an empirical basis (at least from a classical standpoint). This problem have been around for many years, leaving the electromagnetic field (which we have always considered our better understood part of Physics) with an apparent, and rather curious, lack of self-consistency, specially if we compare the situation with its equivalent in General Relativity where the Equation of Geodesics follows both from the underlying Riemannian geometry as well as from Einstein Equations.

During the last decades, the subject seems to have been put aside, as a peculiar feature of the theory rather than a problem. However, its real magnitude comes out when we consider the efforts of Einstein and Weyl (among others), some 80 years ago, of unifying gravitation and electromagnetism in a common geometrical framework. Although one can argue that Einstein and Weyl attempts were far more ambitious than simply obtaining the Lorentz Force, it is also true that Weyl's essential result was the derivation of the electromagnetic field tensor on a geometric basis (We note that his theory, apart of leading to unphysical predictions, also failed to obtain the Lorentz Force)(See [1] and references therein to Weyl's original work).

It is the aim of this article to show that the electromagnetic field tensor and the Lorentz Force are a consequence of the geometric structure of Minkowskian space having its origin in infinitesimal spatial rotations and boosts in space-time. The Lorentz Force is thus also susceptible of a geometrical interpretation and the electromagnetic theory is self-consistent. A heuristic approach to the problem was already published by one of us a few years ago [2]. The scope of this article is to present a rigorous derivation.

II. INTEGRAL CURVES ASSOCIATED WITH THE EQUATIONS OF MOTION AND FINITE LORENTZ TRANSFORMATIONS

In the Minkowskian spacetime M_4, the equations of motion related to a certain particle following an arbitrary trajectory form a second order differential system:

$$
\begin{cases}
\dfrac{dx^\alpha}{d\tau} = u^\alpha \\[2mm]
\dfrac{du^\alpha}{d\tau} = h^\alpha(x, u)
\end{cases}
\tag{1}
$$

where x^α is the position of the particle, u^α is the four-velocity and τ the proper time.

If x_0 is a generic point in spacetime, we write the general integral of (1) as

$$
\begin{cases}
x^\alpha = f^\alpha(x_0, u_0; \tau) \\[2mm]
u^\alpha = \dot{f}^\alpha(x_0, u_0; \tau)
\end{cases}
\tag{2}
$$

$(\cdot \equiv \frac{d}{d\tau})$ such that $f^\alpha(x_0, u_0; 0) \equiv x_0^\alpha$, $\dot{f}^\alpha(x_0, u_0; 0) \equiv u_0^\alpha$.

We thus regard the solutions of (1) as parametric curves in M_4. As the diferential system (1) is autonomous, the integral curves (2) generate a uni-parametric group [3]:

$$
\begin{cases}
f^\alpha[f(x_0, u_0; \tau_1), \dot{f}(x_0, u_0; \tau_1); \tau_2] = f^\alpha(x_0, u_0; \tau_1 + \tau_2) \\[2mm]
\dot{f}^\alpha[f(x_0, u_0; \tau_1), \dot{f}(x_0, u_0; \tau_1); \tau_2] = \dot{f}^\alpha(x_0, u_0; \tau_1 + \tau_2)
\end{cases}
\tag{3}
$$

for any τ_1, τ_2.

Given a spacetime trajectory $x^\alpha = f^\alpha(x_0, u_0; \tau)$ travelled by the particle with a tangent four-velocity vector $u^\alpha(\tau) = \dot{f}^\alpha(x_0, u_0; \tau)$ and adopting a matrix representation of the homogeneous Lorentz Group, we can write:

$$
\dot{f}^\alpha(x_0, u_0; \tau) = L^\alpha_\lambda(x_0; \tau) u_0^\lambda,
\tag{4}
$$

L^α_λ being a matrix of the Lorentz Group which transport the four velocity vector from a point x_0, where the four-velocity is u_0^λ, to another point x, on the curve $x^\alpha = f^\alpha(x_0, u_0; \tau)$, where it is $u^\lambda(\tau)$. (Note that we adopt the "active" point of

3

view and consider L_λ^α as an operator acting on u^λ). Furthermore, as it is readily seen from the second relation in (3) and the last equation, the group condition (3) implies that the τ–parametric family of Lorentz matrices $L_\lambda^\alpha(x_0; \tau)$ verify the relation

$$L(\tau_1) \cdot L(\tau_2) = L(\tau_1 + \tau_2),$$

meaning that this family is an uniparametric subgroup of the Lorentz Group.

In principle, L_λ^α could be any (or any product of them) of the well known matrices which, for completeness, we list below :

$$L_{(1)\lambda}{}^\alpha = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\varphi_1(x_0, \tau) & -\sin\varphi_1(x_0, \tau) \\ 0 & 0 & \sin\varphi_1(x_0, \tau) & \cos\varphi_1(x_0, \tau) \end{pmatrix} \tag{5}$$

$$L_{(2)\lambda}{}^\alpha = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\varphi_2(x_0, \tau) & 0 & \sin\varphi_2(x_0, \tau) \\ 0 & 0 & 1 & 0 \\ 0 & -\sin\varphi_2(x_0, \tau) & 0 & \cos\varphi_2(x_0, \tau) \end{pmatrix} \tag{6}$$

$$L_{(3)\lambda}{}^\alpha = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\varphi_3(x_0, \tau) & -\sin\varphi_3(x_0, \tau) & 0 \\ 0 & 0 & 1 & 0 \\ 0 & \sin\varphi_3(x_0, \tau) & \cos\varphi_3(x_0, \tau) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{7}$$

$$L_{(4)\lambda}{}^\alpha = \begin{pmatrix} \cosh\psi_1(x_o, \tau) & \sinh\psi_1(x_0, \tau) & 0 & 0 \\ \sinh\psi_1(x_0, \tau) & \cosh\psi_1(x_o, \tau) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{8}$$

$$L_{(5)\lambda}{}^\alpha = \begin{pmatrix} \cosh\psi_2(x_o, \tau) & 0 & \sinh\psi_2(x_0, \tau) & 0 \\ 0 & 1 & 0 & 0 \\ \sinh\psi_2(x_0, \tau) & 0 & \cosh\psi_2(x_0, \tau) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{9}$$

$$L_{(6)}{}^{\alpha}{}_{\lambda} = \begin{pmatrix} \cosh \psi_3(x_0, \tau) & 0 & 0 & \sinh \psi_3(x_0, \tau) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh \psi_3(x_0, \tau) & 0 & 0 & \cosh \psi_3(x_0, \tau) \end{pmatrix} \tag{10}$$

$L_{(i)}{}^{\alpha}{}_{\lambda}, (i = 1, 2, 3)$ describe the 3 finite spatial rotations around the Ox^1, Ox^2, Ox^3 axis while the other three remaining are associated to Lorentz boosts along the same axis. $\varphi_i(x_0, \tau)$ and $\psi_i(x_0, \tau)$ being the turning angles with their corresponding arguments.

III. INFINITESIMAL TRANSFORMATIONS, MAXWELL TENSOR AND THE ELECTROMAGNETIC FORCE

As already stated in the introduction, in this section we shall show how infinitesimal transformations in spacetime can be related to our familiar notions of electric and magnetic fields.

We first notice that since the rotation of the spatial part of $u^{\lambda}(\tau)$, between x_0 and x, can proceed at an arbitrary rate, the rotation angle $\varphi_i(x_0, \tau)$ appearing in the first three matrices above can be written as

$$\varphi_i(x_0, \tau) = \bar{b}_i(x_0) \cdot \tau \quad , \quad (i = 1, 2, 3) \tag{11}$$

$\bar{b}_i(x_0)$ meaning the average rotation (around the Ox_i axis) per unit time during the lapse of proper time τ.

In the same way, for the boost "angle" in the other matrices we make

$$\psi_i(x_0, \tau) = \bar{\epsilon}_i(x_0) \cdot \tau, \tag{12}$$

with the same interpretation but this time in terms of space-time "rotations".

We now note that the more general lorentz transformation (which we call $L_G{}^{\alpha}_{\lambda}$) that can act on the four velocity vector is given as the product

$$L_G(x_0; \tau) = \prod_{i=1}^{6} L_{(i)}(x_0; \tau). \tag{13}$$

then, using the preceding parametrization, we can substitute (4) by:

$$\dot{f}^\alpha(x_0; \tau) = L_G{}^\alpha_\lambda(x_0; \tau) u^\lambda_0, \tag{14}$$

Now as $L_G(\tau)$ is an uniparametric subgroup of the Lorentz Group, according to the theory of Lie groups, its derivative, at $\tau = 0$, is an element of its Lie Algebra [3]. Although quite a tedious calculation involving multiplication of the above six matrices (after inserting expressions (11) and (12)), derivation respect to τ and going to the limit $\tau = 0$, it is possible to arrive at the following result:

$$\frac{dL_G{}^\alpha_\lambda}{d\tau}\Big|_{\tau=0} = \begin{pmatrix} 0 & \epsilon_1(x_0) & \epsilon_2(x_0) & \epsilon_3(x_0) \\ \epsilon_1(x_0) & 0 & -b_3(x_0) & b_2(x_0) \\ \epsilon_2(x_0) & b_3(x_0) & 0 & -b_1(x_0) \\ \epsilon_3(x_0) & -b_2(x_0) & b_1(x_0) & 0 \end{pmatrix} \equiv Q^\alpha_\lambda(x_0) \tag{15}$$

where $b_i(x_0)$ and $\epsilon_i(x_0), (i = 1, 2, 3)$ are now the infinitesimal spatial rotation rate and boost rate at x_0.

From Eqs (1), (2) and (14):

$$\dot{u}^\alpha_0 = h^\alpha(x_0, u_0) = \ddot{f}^\alpha(x_0; \tau). \tag{16}$$

Therefore, we get (since x_0 is a generic point)

$$\frac{du^\alpha}{d\tau} = Q^\alpha_\lambda \cdot u^\lambda \quad , \tag{17}$$

This last equation tell us how the structure of Minkowskian spacetime *prescribe a linear evolution* of the four-velocity vector. Since the evolution should actually take place as consequence of the action of an external field, then, up to a proportionality constant, the "rotation" rate ϵ_i can be identified with the i-component of the electric field and the spatial rotation rate b_i with the i-component of the magnetic field:

$$\epsilon_i = k E_i$$

$$b_i = k B_i$$

so that instead of (17) we can write

$$\frac{du^\alpha}{d\tau} = kF^\alpha_\lambda u^\lambda, \tag{18}$$

with F^α_λ the Maxwell field tensor which by virtue of its definition, via Eq. (15), must comply with

$$F^\alpha_\lambda \eta_{\alpha\mu} + F^\alpha_\mu \eta_{\alpha\lambda} = 0, \tag{19}$$

merely a statement of the antisymmetry of $F_{\lambda\mu}$.

Finally, as $F_{\alpha\mu}$ is an element of the Lie Algebra, under a finite Lorentz boost, transform by means of the self-adjoint representation [3], thus giving the right rule of change for every component of both $\vec{E}$ and $\vec{B}$ between inertial frames.

In this new picture of the electromagnetic field, $\vec{E}$ and $\vec{B}$ should be regarded, respectively, as performers of infinitesimal boosts and spatial rotations in spacetime. The new image is a manifestation of the deep harmony between special relativity and electromagnetism.

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